Orthogonality Relations for the "End Problem" for Transversely Isotropic Cylinders

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The differential equations governing the "end problem" in tranversely isotropic cylinders are rewritten as a pair of vector adjoint equations. An orthgonality relation is derived which simplifies the computation of the coefficients in eigenfunction expansions of solutions which decay with distance from the end. Admissible combinations of stress and displacement boundary conditions on the end of the cylinder may be prescribed provided that they lead to solutions which decay away from the end. Conditions which ensure decaying solutions are derived.

Introduction

ALTHOUGH the problem of determining the axisymmetric stress and displacement states near the end of isotropic circular cylinders subjected to self-equilibrating end conditions has been studied extensively,‡ the generalizations to anisotropic elastic materials have been limited by difficulties in integrating the associated differential equations. For transversely isotropic (TI) cylinders Lekhnitskii[2] by means of a stress function approach reduced the problem to the solution of a pair of second-order equations. Eubanks and Sternberg[3] introduced a displacement function which leads to the same formulation. Archer and Byrnes[4] showed that the governing displacement equations can be decoupled to obtain a single equation for the radial displacement. This generalized the results of Swan[5] for the isotropic case.

The only published work dealing with the end problem for the TI case seems to be the work by Warren et al.[7] Using the displacement function of Eubanks and Sternberg[3] they investigate the convergence of the eigenfunction expansions for two particular TI materials and one isotropic case. Byrnes[8] has presented a series solution to the more general end problem where a fully orthotropic material behavior is allowed. Solutions of end problems related to the study of the relief of growth stresses near the end of cross cut timbers are found.

Work on the end problem has been both theoretically and numerically hampered by the lack of convenient orthogonality relations for the various sets of radial eigenfunctions arising from admissible homogeneous boundary conditions at the curved surfaces. Fama[6] has recently derived orthogonality conditions for the isotropic case which are very convenient for the solution of these boundary value problems. A critical review of previous work on orthogonality relations satisfied by sets of eigenfunctions for the isotropic end problem is also given by Fama.[6] In the present work it is shown that an adjoint system can be derived for the TI case which extends the earlier work of Fama.[6] Orthogonality relations which are useful for the solution of boundary value problems with admissible boundary conditions on the flat end are given.

Basic Equations and Analysis

The basic equations for the displacement components $\bar{u}$ and $\bar{w}$ for the axisymmetric TI problem are easily derived by substituting the stress-strain relations

$$\bar{\sigma}_r = b_{11}(\partial\bar{u}/\partial\bar{r}) + b_{12}(\bar{u}/\bar{r}) + b_{13}(\partial\bar{w}/\partial\bar{z}) \tag{1a}$$

$$\bar{\sigma}_\theta = b_{12}(\partial\bar{u}/\partial\bar{r}) + b_{11}(\bar{u}/\bar{r}) + b_{13}(\partial\bar{w}/\partial\bar{z}) \tag{1b}$$

$$\bar{\sigma}_z = b_{13}(\partial\bar{u}/\partial\bar{r}) + b_{13}(\bar{u}/\bar{r}) + b_{33}(\partial\bar{w}/\partial\bar{z}) \tag{1c}$$

$$\bar{\tau}_{rz} = b_{44}(\partial\bar{w}/\partial\bar{r} + \partial\bar{u}/\partial\bar{z}) \tag{1d}$$

into the equilibrium equations to obtain

$$\frac{\partial^2\bar{u}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{u}}{\partial\bar{r}} - \frac{\bar{u}}{\bar{r}^2} + \frac{b_{44}}{b_{11}}\frac{\partial^2\bar{u}}{\partial\bar{z}^2} + \frac{(b_{13}+b_{44})}{b_{11}}\frac{\partial^2\bar{w}}{\partial\bar{r}\,\partial\bar{z}} = 0 \tag{2a}$$

$$\frac{\partial^2\bar{w}}{\partial\bar{r}^2} + \frac{1}{r}\frac{\partial\bar{w}}{\partial\bar{r}} + \frac{b_{33}}{b_{44}}\frac{\partial^2\bar{w}}{\partial\bar{z}^2} + \frac{(b_{13}+b_{44})}{b_{44}}\left(\frac{\partial^2\bar{u}}{\partial\bar{r}\,\partial\bar{z}} + \frac{1}{r}\frac{\partial\bar{u}}{\partial\bar{z}}\right) = 0 \tag{2b}$$

where $\bar{r}$ and $\bar{z}$ are the radial and axial coordinates, and the b_{ij} are the TI elastic constants.

For a given set of eigenfunctions $u_k(r)$ and $w_k(r)$ corresponding to a particular pair of homogeneous boundary conditions at the outer radius $\bar{r} = b$, we seek solutions in the form

$$\bar{u}/b = \sum_k a_k u_k e^{-\lambda_k z} \tag{3a}$$

$$\bar{w}/b = \sum_k a_k w_k e^{-\lambda_k z} \tag{3b}$$

where the nondimensional coordinates $r = \bar{r}/b$ and $z = \bar{z}/b$ are defined and the λ_k's are the eigenvalues. Using Eq. (2) it follows that

$$u_k'' + \frac{u_k'}{r} - \frac{u_k}{r^2} + \frac{b_{44}}{b_{11}}\lambda_k{}^2 u_k - \lambda_k\frac{(b_{13}+b_{44})}{b_{11}}w_k' = 0 \tag{4a}$$

$$w_k'' + \frac{w_k'}{r} + \frac{b_{33}}{b_{44}}\lambda_k{}^2 w_k - \lambda_k\frac{(b_{13}+b_{44})}{b_{44}}\left(u_k' + \frac{u_k}{r}\right) = 0 \tag{4b}$$

where

$$(\ )' = d/dr(\ )$$

Nondimensional forms for the stresses $\bar{\sigma}_z$ and $\bar{\tau}_{rz}$ follow from

$$\bar{\tau}_{rz}/b_{44} = \sum_k a_k \tau_k(r) e^{-\lambda_k z} \tag{5a}$$

$$\bar{\sigma}_z/b_{44} = \sum_k a_k \sigma_k(r) e^{-\lambda_k z} \tag{5b}$$

and thus

$$\tau_k = w_k' - \lambda_k u_k \tag{6a}$$

$$\sigma_k = \frac{b_{13}}{b_{44}}\left(u_k' + \frac{u_k}{r}\right) - \frac{b_{33}}{b_{44}}\lambda_k w_k \tag{6b}$$

Following the approach used by Fama,[6] a pair of equations for either w_k and τ_k or u_k and σ_k may be derived so as to retain only terms in $\lambda_k{}^2$. This is an important step in obtaining orthogonality relations. By introducing the auxiliary variable

$$\phi_k = \alpha w_k' + \beta\tau_k \tag{7}$$

and seeking a linear combination of

$$A(d/dr)(\text{Eq. 4b}) + B\lambda_k(\text{Eq. 4a}) \tag{8}$$

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‡ See Lure[1] for references to earlier work.

such that Eq. (8) takes the form

$$\phi_k'' + \frac{\phi_k'}{r} - \frac{\phi_k}{r^2} + \lambda_k^2 s^2 \phi_k = 0 \tag{9}$$

an eigenvalue problem for s^2 results. That is,

$$\begin{bmatrix} \dfrac{b_{33}}{b_{44}} & -\dfrac{(b_{13}+b_{44})}{b_{11}} \\ 0 & \dfrac{b_{44}}{b_{11}} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = s^2 \begin{bmatrix} 1 & 0 \\ -\dfrac{(b_{13}+b_{44})}{b_{44}} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{10}$$

and nontrivial solutions for A and B are possible only if s_1^2 and s_2^2 are roots of

$$s^4 - s^2\left[\frac{b_{33}}{b_{44}} + \frac{b_{44}}{b_{11}} - \frac{(b_{13}+b_{44})^2}{b_{11}b_{44}}\right] + \frac{b_{33}}{b_{11}} = 0 \tag{11}$$

These roots are the reciprocals§ of the roots derived by Lekhnitskii[2] as "separation constants" in the factorization of the governing equations into two second-order equations.

Lekhnitskii[2] proves that "s_1 and s_2 can be only real or complex but cannot be purely imaginary." Warren et al.[7] further state that from the positive definiteness of the strain energy function, Eq. (11) can have no negative or vanishing roots. They find real roots in the case of magnesium and complex conjugates in the case of zinc. Archer and Byrnes[4] found real roots for particular species of Douglas fir, spruce, and pine.

By defining the vector

$$\bar{\zeta}_k^{(2)} = r^{1/2}\begin{bmatrix} w_k \\ \dfrac{Ew_k'}{s_2^2} - \dfrac{R}{s_2^2}\tau_k \end{bmatrix} = \begin{bmatrix} \zeta_{1k} \\ \zeta_{2k} \end{bmatrix} \tag{12}$$

where

$$R = (b_{13}+b_{44})/b_{11} \tag{13}$$

and

$$E = s_2^2 + \frac{b_{13}}{b_{11}} = -s_1^2 + \frac{b_{33}}{b_{44}} - \frac{b_{13}}{b_{44}}R \tag{14}$$

Eqs. (4b) and (9) may be written as

$$L\bar{\zeta}_k^{(2)} + \lambda_k^2 \bar{H}\bar{\zeta}_k^{(2)} = 0 \tag{15}$$

where

$$L\bar{\zeta}_k^{(2)} = \begin{bmatrix} \dfrac{d^2\zeta_{1k}}{dr^2} + \dfrac{\zeta_{1k}}{4r^2} - \dfrac{d\zeta_{2k}}{dr} - \dfrac{\zeta_{2k}}{2r} \\ \dfrac{d^2\zeta_{2k}}{dr^2} - \dfrac{3}{4}\dfrac{\zeta_{2k}}{r^2} \end{bmatrix} \tag{16}$$

and

$$\bar{H} = \begin{bmatrix} s_1^2 & 0 \\ 0 & s_2^2 \end{bmatrix} \tag{17}$$

By means of an analogous derivation, an adjoint system of equations follows by defining

$$\bar{\eta}_k^{(1)} = r^{1/2}\begin{bmatrix} \dfrac{E}{s_1^2}\left(u_k' + \dfrac{u_k}{r}\right) + \dfrac{R}{s_1^2}\sigma_k \\ u_k \end{bmatrix} = \begin{bmatrix} \eta_{1k} \\ \eta_{2k} \end{bmatrix} \tag{18}$$

and thus combining Eqs. (4a) and the counterpart of Eq. (9) to obtain

$$L^*\bar{\eta}_k^{(1)} + \lambda_k^2 \bar{H}\bar{\eta}_k^{(1)} = 0 \tag{19}$$

where

$$L^*\bar{\eta}_k^{(1)} = \begin{bmatrix} \dfrac{d^2\eta_{1k}}{dr^2} + \dfrac{\eta_{1k}}{4r^2} \\ \dfrac{d^2\eta_{2k}}{dr^2} - \dfrac{3}{4}\dfrac{\eta_{2k}}{r^2} + \dfrac{d\eta_{1k}}{dr} - \dfrac{\eta_{1k}}{2r} \end{bmatrix} \tag{20}$$

§ See Archer and Byrnes[4] for a discussion of this point.

Derivation of the Orthogonality Relation

Now because L and L^* are adjoint operators

$$\int_a^1 \{L\bar{\zeta}_k^{(2)}\bar{\eta}_j^{(1)} - L^*\bar{\eta}_j^{(1)}\bar{\zeta}_k^{(2)}\}\,dr = [\eta_{1j}\zeta_{1k}' - \eta_{1j}\zeta_{2k} + \eta_{2j}\zeta_{2k}' - \zeta_{1k}\eta_{1j}' - \zeta_{2k}\eta_{2j}']_a^1 = [\text{boundary terms}]_a^1 = (\lambda_j^2 - \lambda_k^2)\int_a^1 \bar{\zeta}_k^{(2)}\bar{H}\bar{\eta}_j^{(1)}\,dr \tag{21}$$

where a is the nondimensional inner radius of the cylinder.

The orthogonality relation

$$\text{(A)} \qquad \int_a^1 \bar{\zeta}_k^{(2)}\bar{H}\bar{\eta}_j^{(1)}\,dr = 0 \qquad \text{for} \qquad k \neq j$$

holds whenever the expression denoted by $[\text{boundary terms}]_a^1$ vanishes. A second important relation may be derived starting with the identity

$$\int_a^1 \bar{\zeta}_k^{(2)}\bar{H}\bar{\eta}_j^{(1)}\,dr = E[rw_k u_j]_a^1 + R\int_a^1 (w_k\sigma_j - \tau_k u_j)r\,dr \tag{22}$$

and noting that

$$\text{(B)} \qquad \int_a^1 (w_k\sigma_j - \tau_k u_j)r\,dr = 0 \qquad \text{for} \qquad k \neq j$$

holds whenever

$$[\text{boundary terms}]_a^1 = (\lambda_j^2 - \lambda_k^2)E[\zeta_{1k}\eta_{2j}]_a^1 \tag{23}$$

It may be noted that for the special case of isotropic materials $s_1^2 = s_2^2 = 1$, and $\bar{H}$ reduces to the identity matrix and orthogonality relation of Fama is recovered. The definitions of the vectors $\bar{\zeta}_k^{(2)}$ and $\bar{\eta}_k^{(1)}$ reduce to those given by Fama except for a scale factor.

Homogeneous Boundary Conditions

The restriction of admissible homogeneous boundary conditions on the curved surfaces leads to prescribing either $\bar{u}$ or $\bar{\sigma}_r$ together with either $\bar{w}$ or $\bar{\tau}_{rz}$ on the surfaces $r = a, 1$.

$$\bar{w} = 0 \text{ implies} \begin{cases} \zeta_{1k} = 0 \\ \dfrac{d}{dr}\eta_{2j} + \dfrac{\eta_{2j}}{2r} - \dfrac{1}{F}\eta_{1j} = 0 \end{cases}$$

$$\bar{u} = 0 \text{ implies} \begin{cases} \dfrac{d}{dr}\zeta_{1k} - \dfrac{\zeta_{1k}}{2r} - \dfrac{b_{11}s_2^2}{b_{44}F}\zeta_{2k} = 0 \\ \eta_{2j} = 0 \end{cases}$$

$$\bar{\tau}_{rz} = 0 \text{ implies} \begin{cases} \dfrac{d}{dr}\zeta_{1k} - \dfrac{\zeta_{1k}}{2r} - \dfrac{s_2^2}{E}\zeta_{2k} = 0 \\ \dfrac{d}{dr}\eta_{1j} - \dfrac{\eta_{1j}}{2r} + \lambda_j^2 E\eta_{2j} = 0 \end{cases}$$

$$\bar{\sigma}_r = 0 \text{ implies} \begin{cases} \dfrac{d}{dr}\zeta_{2k} - C\dfrac{\zeta_{2k}}{2r} + D\left(\dfrac{1}{r}\dfrac{d}{dr}\zeta_{1k} - \dfrac{\zeta_{1k}}{2r^2}\right) + \lambda_k^2 E\zeta_{1k} = 0 \\ \dfrac{d}{dr}\eta_{2j} - \dfrac{B}{A}\dfrac{\eta_{2j}}{2r} + \dfrac{1}{A}\eta_{1j} = 0 \end{cases}$$

where

$$A = \frac{(s_2^2 b_{11} + b_{13})}{b_{13}}, \qquad F = \frac{(s_2^2 b_{11} - b_{44})}{b_{44}}$$

$$C = -2s_2^2\frac{(b_{12} - b_{11})}{b_{44}} - 1, \qquad D = \frac{(b_{11} - b_{12})}{b_{11}}F$$

$$B = \frac{2(b_{11} - b_{12})s_2^2(b_{13} + b_{44})}{b_{13}b_{44}} - A$$

Case 1

Prescribing $\bar{u} = \bar{w} = 0$ on $r = a, 1$ requires

$$\zeta_{1k} = 0, \quad \frac{d}{dr}\zeta_{1k} - \frac{b_{11}s_2^2}{b_{44}F}\zeta_{2k} = 0 \tag{24a}$$

$$\eta_{2j} = 0, \quad \frac{d}{dr}\eta_{2j} - \frac{1}{F}\eta_{1j} = 0 \tag{24b}$$

Substituting Eqs. (24a) and (24b) into Eq. (23) gives [boundary terms]$_a^1 = 0$ and $[\zeta_{1k}\eta_{2j}]_a^1 = 0$ therefore the relations (A) and (B) both hold.

Case 2

Prescribing $\bar{u} = \bar{\tau}_{rz} = 0$ on $r = a$, 1 requires

$$\zeta_{2k} = 0, \quad \frac{d\zeta_{1k}}{dr} - \frac{\zeta_{1k}}{2r} = 0 \tag{25a}$$

$$\eta_{2j} = 0, \quad \frac{d}{dr}\eta_{1j} - \frac{\eta_{1j}}{2r} = 0 \tag{25b}$$

and relations (A) and (B) both hold.

Case 3

Prescribing $\bar{w} = \bar{\sigma}_r = 0$ on $r = a$, 1 requires

$$\zeta_{1k} = 0, \quad \frac{d}{dr}\zeta_{2k} - C\frac{\zeta_{2k}}{2r} + D\left(\frac{1}{r}\frac{d}{dr}\zeta_{1k}\right) = 0 \tag{26a}$$

$$\frac{\eta_{2j}}{r} - \frac{b_{11}}{(b_{11}-b_{12})F}\eta_{1j} = 0, \quad \frac{d}{dr}\eta_{2j} - \frac{(b_{11}-2b_{12})}{2(b_{11}-b_{12})F}\eta_{1j} = 0 \tag{26b}$$

and relations (A) and (B) both hold.

Case 4

Prescribing $\bar{\tau}_{rz} = \bar{\sigma}_r = 0$ on $r = a$, 1 requires

$$\frac{d}{dr}\zeta_{1k} - \frac{\zeta_{1k}}{2r} - \frac{s_2{}^2}{E}\zeta_{2k} = 0, \quad \frac{d}{dr}\zeta_{2k} + \frac{\zeta_{2k}}{r}\left(\frac{s_2{}^2 D}{E} - \frac{C}{2}\right) + \lambda_k{}^2 E\zeta_{1k} = 0 \tag{27a}$$

$$\frac{d}{dr}\eta_{2j} - \frac{B}{A}\frac{\eta_{2j}}{2r} + \frac{1}{A}\eta_{1j} = 0, \quad \frac{d}{dr}\eta_{1j} - \frac{\eta_{1j}}{2r} + \lambda_j{}^2 E\eta_{2j} = 0 \tag{27b}$$

Here only relation (B) holds.

Boundary-Value Problems

1) Prescribe $\bar{\sigma}_z/b_{44} = \sigma_o(r)$, $\bar{u}/b = u_o(r)$ on $z = 0$ and conditions 1–4 on $r = a$, 1. With the assumption that the system of eigenfunctions $\{\bar{\eta}_k\}$ is complete for cases 1–4, let

$$\bar{\eta}_o(r) = \sum_k a_k \bar{\eta}_k \tag{28}$$

where

$$\bar{\eta}_o(r) = \begin{bmatrix}\eta_{o1}\\ \eta_{o2}\end{bmatrix} = r^{1/2}\left[\begin{Bmatrix}\frac{E}{s_1{}^2}\left(u_o' + \frac{u_o}{r}\right) + \frac{R}{s_1{}^2}\sigma_o\\ u_o\end{Bmatrix}\right] \tag{29}$$

Applying the regular expansion technique to Eq. (28) gives

$$\int_a^1 \bar{H}\bar{\eta}_o\bar{\zeta}_j dr - \sum_k a_k E[\eta_{2k}\zeta_{1k}]_a^1 = \sum_k a_k\left\{\int_a^1 \bar{H}\bar{\eta}_k\bar{\zeta}_j dr - E[\eta_{2k}\zeta_{1k}]_a^1\right\} \tag{30}$$

where $\bar{\zeta}_j$ is an arbitrary eigenmode corresponding to λ_j. Next, incorporating relation (B) and the definition of $\bar{\eta}_o(r)$, Eq. (30) reduces to

$$a_k = M_k{}^{-1}\int_a^1 \bar{H}\bar{\eta}_o\bar{\zeta}_k dr - E[\eta_{02}\zeta_{1k}] \tag{31}$$

where

$$M_k = \int_a^1 \bar{H}\bar{\eta}_k\bar{\zeta}_k dr - E[\eta_{2k}\zeta_{1k}]_a^1 = R\int_a^1 (\sigma_k w_k - u_k\tau_k) r\, dr$$

Finally, the property of uniform convergence of Eq. (28), as a consequence of the assumption of completeness, gives

$$a_k = M_k{}^{-1}\int_a^1 (\sigma_o w_k - u_o\tau_k) r\, dr \tag{32}$$

Thus for each unique set of eigenvalues and eigenfunctions obtained from conditions 1–4 on $r = a$, 1, the solution to the present boundary-value problem is

$$\bar{E}(r,z) = \sum_k a_k\bar{\chi}_k(r) e^{-\lambda_k z} \tag{33}$$

where

$$\bar{\chi}_k(r) = \begin{bmatrix}\bar{\phi}_k(r)\\ \bar{\psi}_k(r)\end{bmatrix} = \begin{bmatrix}\sigma_k(r)\\ u_k(r)\\ w_k(r)\\ -\tau_k(r)\end{bmatrix} \tag{34}$$

with

$$a_k = M_k{}^{-1}\int_a^1 \bar{\phi}_o(r)\bar{\psi}_k(r) r\, dr \tag{35}$$

and

$$\bar{\phi}_o(r) = \begin{bmatrix}\sigma_o(r)\\ u_o(r)\end{bmatrix} \tag{36}$$

2) Prescribe $\bar{\tau}_{rz}/b_{44} = \tau_o(r)$, $\bar{w}/b = w_o(r)$ on $z = 0$ and conditions 1–4 on $r = a$, 1. The second boundary-value problem can be solved using a similar technique with the eigenfunctions $\{\bar{\zeta}_k\}$ to give

$$\bar{E}(r,z) = \sum_k b_k\bar{\chi}(r) e^{-\lambda_k z} \tag{37}$$

with

$$b_k = M_k{}^{-1}\int_a^1 \bar{\psi}_o\bar{\phi}_k r\, dr \tag{38}$$

where

$$\bar{\psi}_o(r) = \begin{bmatrix}w_o(r)\\ -\tau_o(r)\end{bmatrix} \tag{39}$$

3) Prescribe $\bar{\sigma}_z/b_{44} = \sigma_o(r)$, $\bar{\tau}_{rz}/b_{44} = \tau_o(r)$ on $z = 0$ and conditions 1–4 on $r = a$, 1. This problem differs from cases 1 and 2 in that the known prescribed tractions are not given directly by either the $\{\bar{\zeta}_k\}$ or $\{\bar{\eta}_k\}$ set of eigenfunctions. Therefore the technique to calculate the expansion coefficients directly must be modified.

A unique solution for $\bar{u}$ and $\bar{w}$ on $z = 0$ is given by

$$\bar{u}(r,0)/b = u_o(r), \quad \bar{w}(r,0)/b = w_o(r) \tag{40}$$

where

$$\sum_j a_j u_j = u_o \quad \text{and} \quad \sum_j a_j w_j = w_o$$

Expanding $\bar{\eta}_o(r)$ and $\bar{\zeta}_o(r)$ in terms of the $\{\bar{\eta}_k\}$ and $\{\bar{\zeta}_k\}$ set of eigenfunctions, respectively, leads to

$$a_k = \tfrac{1}{2}M_k{}^{-1}\int_a^1 (\sigma_o w_k - \tau_o u_k + w_o\sigma_k - u_o\tau_k) r\, dr \tag{41}$$

where u_o and w_o are defined by Eq. (40). Reference 6 contains a proof that asserts

$$\sum_k a_k\sigma_k = \sigma_o \quad \text{and} \quad \sum_k a_k\tau_k = \tau_o$$

implying the unique solution to the present boundary-value problem is

$$\bar{E}(r,z) = \sum_k a_k\bar{\chi}_k(r) e^{-\lambda_k z} \tag{42}$$

where

$$a_k = \tfrac{1}{2}M_k{}^{-1}\int_a^1 \left\{\sigma_o w_k - \tau_o u_k + \sum_j a_j(w_j\sigma_k - u_j\tau_k)\right\} r\, dr \tag{43}$$

4) Prescribe $\bar{u}/b = u_o(r)$, $\bar{w}/b = w_o(r)$ on $z = 0$ and conditions 1–4 on $r = a$, 1. Boundary-value problem 4 is treated in a similar manner to give

$$\bar{E}(r,z) = \sum_k b_k\bar{\chi}_k(r) e^{-\lambda_k z} \tag{44}$$

where

$$b_k = \tfrac{1}{2}M_k{}^{-1}\int_a^1 \left\{w_o\sigma_k - u_o\tau_k + \sum_j b_j(\sigma_j w_k - \tau_j u_k)\right\} r\, dr \tag{45}$$

In summary, the coefficients of the eigenfunction expansions for boundary-value problem 1 and 2 can be calculated directly, whereas 3 and 4 lead to infinite sets of algebraic equations for the coefficients.

Conditions for Decaying Solutions

For the study of end effects in cylinders only solutions which decay with distance from the plane end (e.g., eigenfunction solutions for $Re\,\lambda_k > 0$) are of interest. However, the exclusion of the eigenfunction corresponding to $\lambda = 0$, the rigid body translation and uniform axial stress solution, produces certain constraints on the problem formulation. These conditions under which decaying solutions exist can be obtained by requiring the numerator of the expansion coefficient for $\lambda = 0$ to be identically zero.

The plane deformation solution can be written as

$$\bar{E}^*(r, z) = a^*\bar{\chi}(r) \quad (46)$$

where $\bar{\chi}^*(r)$ is obtained from Eqs. (15) and (19) with $\lambda_k = 0$ and a^* is the corresponding expansion coefficient. Thus

$$\bar{\zeta}^* = \begin{bmatrix} \alpha_1 r^{5/2}(\frac{1}{2} - \ln r) + \beta_2 r^{1/2} + \frac{1}{2}\beta_1 r^{1/2} \ln r \\ \alpha_1 r^{3/2} + \alpha_2 r^{-1/2} \end{bmatrix} \quad (47)$$

and

$$\bar{\eta}^* = \begin{bmatrix} \gamma_1 r^{1/2} \ln r + \gamma_2 r^{1/2} \\ \gamma_1 r^{5/2}/3 + \delta_1 r^{3/2} + \delta_2 r^{-1/2} \end{bmatrix} \quad (48)$$

with α_i, β_i, γ_i, and δ_i arbitrary constants determined from the curved surface boundary conditions.

1) If $\sigma_o(r)$ and $u_o(r)$ are prescribed on $z = 0$, the required constraint is given by

$$a^* = M^{*-1} \int_a^1 (\sigma_o w^* - u_o \tau^*) r\,dr = 0$$

In cases 1 and 2 $\bar{\zeta}^* = 0$ which leads to $w^* = \tau^* = 0$ provided

$$\frac{a^2 \ln a}{(1-a^2)} \neq \frac{s_2{}^2 b_{11}}{2b_{44}F} \quad \text{in case 1}$$

and

$$\frac{a^2 \ln a}{(1-a^2)} \neq s_2{}^2 \frac{(b_{11} - b_{12})}{2b_{44}} - 1 \quad \text{in case 3}$$

Thus $\lambda = 0$ is not an eigenvalue and there are no constraints on $\sigma_o(r)$ and $u_o(r)$.

In cases 2 and 4 $w^* = \beta_2$, $\tau^* = 0$ provided $(1-a)/2a \ln a \neq E$ in case 4, which gives

$$\int_a^1 \sigma_o(r) r\,dr = 0 \quad (49)$$

as the necessary condition for the existence of decaying solutions.

2. If $w_o(r)$ and $\tau_o(r)$ are prescribed on $z = 0$, we require $a^* = M^{*-1} \int_a^1 (w_o \sigma^* - \tau_o u^*) r\,dr = 0$. In cases 1 and 3 $\bar{\eta}^* = 0$ and $a^* = 0$ without constraints on $w_o(r)$ and $\tau_o(r)$ provided

$$\frac{1na}{(a-1)} \neq \frac{(s_2{}^2 b_{11} - b_{44})}{b_{44}} \quad \text{in case 1}$$

and

$$\frac{\ln a}{(a-1)} \neq \frac{2F(b_{12} - b_{11})}{3b_{12}} \quad \text{in case 3}$$

In case 2 $\sigma^* = \delta_2'$, $u^* = 0$ which gives

$$\int_a^1 w_o(r) r\,dr = 0 \quad (50)$$

as the required constraint placed on $w_o(r)$. In case 4

$$u^* = \frac{2r}{(B-3A)}\gamma_2, \quad \sigma^* = \frac{1}{R}\left(s_1{}^2 - \frac{4E}{(B-3A)}\right)\gamma_2$$

requiring

$$\int_a^1 \{(s_1{}^2[B-3A] - 4A) w_o - 2Rr\tau_o\} r\,dr = 0 \quad (51)$$

to be satisfied by $w_o(r)$ and $\tau_o(r)$.

3. If $\sigma_o(r)$ and $\tau_o(r)$ are prescribed on $z = 0$, the condition

$$a^* = \tfrac{1}{2} M^{*-1} \int_a^1 \left\{ \sigma_o w^* - \tau_o u^* + \sum_j a_j (w_j \sigma^* - u_j \tau^*) \right\} r\,dr = 0$$

is satisfied for case 2 by requiring

$$\int_a^1 \sigma_o(r) r\,dr = 0 \quad (52)$$

Since displacements are not included in the boundary conditions for case 4, the exclusion of rigid body displacements leads directly to Eq. (52) without consideration of the expansion coefficient.

4. If $u_o(r)$ and $w_o(r)$ are prescribed on $z = 0$, the required constraint is given by

$$b^* = \tfrac{1}{2} M^{*-1} \int_a^1 \left\{ w_o \sigma^* - u_o \tau^* - \sum_j b_j (\sigma_j w^* - \tau_j u^*) \right\} r\,dr = 0$$

In this problem a uniform axial displacement becomes indeterminate, since $\sum_k a_k w_k$ converges to $w_o(r) + \alpha$, provided Eq. (50) or (51) is satisfied but the expansion coefficient is independent of α (e.g. an integration of the equilibrium equation with respect to r gives $\int_a^1 \sigma_j r\,dr = 0$ for each eigenmode). Consequently, the value of α must be such that Eqs. (50) or (51) is not violated. Thus, in case 2

$$\alpha = \frac{-2}{(1-a^2)} \int_a^1 w_o(r) r\,dr \quad (53)$$

and for case 4

$$\alpha = \frac{-2}{(1-a^2)} \int_a^1 w_o(r) r\,dr + \frac{4R}{(1-a^2)Q} \sum_k b_k \int_a^1 \tau_k r^2\,dr \quad (54)$$

where $Q = s_1{}^2(B-3A) - 4A$, are the necessary and sufficient conditions for the existence of decaying solutions for 4.

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